Section A: Pure Mathematics

1 A number of the form $1/N$, where N is an integer greater than 1, is called a *unit fraction.*

Noting that
$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} \quad \text{and} \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{12},$$
guess a general result of the form
$$\frac{1}{N} = \frac{1}{a} + \frac{1}{b} \qquad (*)$$
and hence prove that any unit fraction can be expressed as the sum of two distinct unit fractions.

By writing $(*)$ in the form
$$(a-N)(b-N) = N^2$$
and by considering the factors of N^2, show that if N is prime, then there is only one way of expressing $1/N$ as the sum of two distinct unit fractions.

Prove similarly that any fraction of the form $2/N$, where N is prime number greater than 2, can be expressed uniquely as the sum of two distinct unit fractions.

2 Prove that if $(x-a)^2$ is a factor of the polynomial $\mathrm{p}(x)$, then $\mathrm{p}'(a) = 0$. Prove a corresponding result if $(x-a)^4$ is a factor of $\mathrm{p}(x)$.

Given that the polynomial
$$x^6 + 4x^5 - 5x^4 - 40x^3 - 40x^2 + 32x + k$$
has a factor of the form $(x-a)^4$, find k.

3 The lengths of the sides BC, CA, AB of the triangle ABC are denoted by a, b, c, respectively. Given that
$$b = 8 + \epsilon_1,\ c = 3 + \epsilon_2,\ A = \pi/3 + \epsilon_3,$$
where ϵ_1, ϵ_2, and ϵ_3 are small, show that $a \approx 7 + \eta$, where $\eta = (13\,\epsilon_1 - 2\,\epsilon_2 + 24\sqrt{3}\,\epsilon_3)/14$.

Given now that
$$|\epsilon_1| \leqslant 2 \times 10^{-3}, \quad |\epsilon_2| \leqslant 4\cdot 9 \times 10^{-2}, \quad |\epsilon_3| \leqslant \sqrt{3} \times 10^{-3},$$
find the range of possible values of η.

4 Prove that

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

and that, for every positive integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

By considering $(5-i)^2(1+i)$, or otherwise, prove that

$$\arctan(7/17) + 2\arctan(1/5) = \pi/4.$$

Prove also that

$$3\arctan(1/4) + \arctan(1/20) + \arctan(1/1985) = \pi/4.$$

[Note that $\arctan\theta$ is another notation for $\tan^{-1}\theta$.]

5 It is required to approximate a given function $\mathrm{f}(x)$, over the interval $0 \leqslant x \leqslant 1$, by the linear function λx, where λ is chosen to minimise

$$\int_0^1 \Big(\mathrm{f}(x) - \lambda x\Big)^2 \,\mathrm{d}x.$$

Show that

$$\lambda = 3\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x.$$

The residual error, R, of this approximation process is such that

$$R^2 = \int_0^1 \Big(\mathrm{f}(x) - \lambda x\Big)^2 \,\mathrm{d}x.$$

Show that

$$R^2 = \int_0^1 \Big(\mathrm{f}(x)\Big)^2 \,\mathrm{d}x - \frac{1}{3}\lambda^2.$$

Given now that $\mathrm{f}(x) = \sin(\pi x/n)$, show that (i) for large n, $\lambda \approx \pi/n$ and (ii) $\lim_{n\to\infty} R = 0$.

Explain why, prior to any calculation, these results are to be expected.

[You may assume that, when θ is small, $\sin\theta \approx \theta - \theta^3/6$ and $\cos\theta \approx 1 - \theta^2/2$.]

6 Show that

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \frac{1+\cos\theta}{\sin\theta} = \tan(\pi/2 - \theta/2),$$

where $t = \tan(\theta/2)$.

Use the substitution $t = \tan(\theta/2)$ to show that, for $0 < \alpha < \pi/2$,

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin\theta}\,\mathrm{d}\theta = \frac{\alpha}{\sin\alpha},$$

and deduce a similar result for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\alpha\cos\theta}\,\mathrm{d}\theta.$$

7 The line l has vector equation $\mathbf{r} = \lambda\mathbf{s}$, where

$$\mathbf{s} = (\cos\theta + \sqrt{3}\,)\,\mathbf{i} + (\sqrt{2}\ \sin\theta)\,\mathbf{j} + (\cos\theta - \sqrt{3}\,)\,\mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between l and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between l and m is $\pi/6$.

A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

8 (i) Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\,\mathrm{e}^{-x^2}(y+3)^{\frac{1}{2}} = 0 \qquad (x \geqslant 0),$$

that satisfies the condition $y = 6$ when $x = 0$. Find y in terms of x and show that $y \to 1$ as $x \to \infty$.

(ii) Let y be any solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x\,\mathrm{e}^{6x^2}(y+3)^{1-k} = 0 \qquad (x \geqslant 0).$$

Find a value of k such that, as $x \to \infty$, $\mathrm{e}^{-3x^2}y$ tends to a finite non-zero limit, which you should determine.

Section B: Mechanics

9 In an aerobatics display, Jane and Karen jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v, Jane experiences air resistance kv per unit mass but Karen, who spread-eagles, experiences air resistance $kv + (2k^2/g)v^2$ per unit mass. Show that Jane's speed can never reach g/k. Obtain the corresponding result for Karen.

Jane opens her parachute when her speed is $g/(3k)$. Show that she has then been in free fall for time $k^{-1}\ln(3/2)$.

Karen also opens her parachute when her speed is $g/(3k)$. Find the time she has then been in free fall.

10 A long light inextensible string passes over a fixed smooth light pulley. A particle of mass 4 kg is attached to one end A of this string and the other end is attached to a second smooth light pulley. A long light inextensible string BC passes over the second pulley and has a particle of mass 2 kg attached at B and a particle of mass of 1 kg attached at C. The system is held in equilibrium in a vertical plane. The string BC is then released from rest. Find the accelerations of the two moving particles.

After T seconds, the end A is released so that all three particles are now moving in a vertical plane. Find the accelerations of A, B and C in this second phase of the motion. Find also, in terms of g and T, the speed of A when B has moved through a total distance of $0.6gT^2$ metres.

11 The string AP has a natural length of $1{\cdot}5$ metres and modulus of elasticity equal to $5g$ newtons. The end A is attached to the ceiling of a room of height $2{\cdot}5$ metres and a particle of mass $0{\cdot}5$ kg is attached to the end P. The end P is released from rest at a point $0{\cdot}5$ metres above the floor and vertically below A. Show that the string becomes slack, but that P does not reach the ceiling.

Show also that while the string is in tension, P executes simple harmonic motion, and that the time in seconds that elapses from the instant when P is released to the instant when P first returns to its original position is

$$\left(\frac{8}{3g}\right)^{\frac{1}{2}} + \left(\frac{3}{5g}\right)^{\frac{1}{2}} \Big(\pi - \arccos(3/7)\Big).$$

[Note that $\arccos x$ is another notation for $\cos^{-1} x$.]

Section C: Probability and Statistics

12 *Tabulated values of $\Phi(\cdot)$, the cumulative distribution function of a standard normal variable, should not be used in this question.*

Henry the commuter lives in Cambridge and his working day starts at his office in London at 0900. He catches the 0715 train to King's Cross with probability p, or the 0720 to Liverpool Street with probability $1 - p$. Measured in minutes, journey times for the first train are $N(55, 25)$ and for the second are $N(65, 16)$. Journey times from King's Cross and Liverpool Street to his office are $N(30, 144)$ and $N(25, 9)$, respectively. Show that Henry is more likely to be late for work if he catches the first train.

Henry makes M journeys, where M is large. Writing A for $1 - \Phi(20/13)$ and B for $1 - \Phi(2)$, find, in terms of A, B, M and p, the expected number, L, of times that Henry will be late and show that for all possible values of p,

$$BM \leqslant L \leqslant AM.$$

Henry noted that in 3/5 of the occasions when he was late, he had caught the King's Cross train. Obtain an estimate of p in terms of A and B.

[A random variable is said to be $N\left(\mu, \sigma^2\right)$ if it has a normal distribution with mean μ and variance σ^2.]

13 A group of biologists attempts to estimate the magnitude, N, of an island population of voles (*Microtus agrestis*). Accordingly, the biologists capture a random sample of 200 voles, mark them and release them. A second random sample of 200 voles is then taken of which 11 are found to be marked. Show that the probability, p_N, of this occurrence is given by

$$p_N = k\frac{\big((N-200)!\big)^2}{N!(N-389)!},$$

where k is independent of N.

The biologists then estimate N by calculating the value of N for which p_N is a maximum. Find this estimate.

All unmarked voles in the second sample are marked and then the entire sample is released. Subsequently a third random sample of 200 voles is taken. Write down the probability that this sample contains exactly j marked voles, leaving your answer in terms of binomial coefficients.

Deduce that

$$\sum_{j=0}^{200}\binom{389}{j}\binom{3247}{200-j} = \binom{3636}{200}.$$

14 The random variables $X_1,\ X_2,\ \ldots\ ,\ X_{2n+1}$ are independently and uniformly distributed on the interval $0 \leqslant x \leqslant 1$. The random variable Y is defined to be the median of $X_1,\ X_2,\ \ldots\ ,\ X_{2n+1}$. Given that the probability density function of Y is $\mathrm{g}(y)$, where

$$\mathrm{g}(y) = \begin{cases} ky^n(1-y)^n & \text{if } 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

use the result

$$\int_0^1 y^r(1-y)^s\,dy = \frac{r!s!}{(r+s+1)!}$$

to show that $k = (2n+1)!/(n!)^2$, and evaluate $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. Hence show that, for any given positive number d, the inequality

$$\mathrm{P}\left(|Y - 1/2| < d/\sqrt{n}\right) < \mathrm{P}\left(|\bar{X} - 1/2| < d/\sqrt{n}\right)$$

holds provided n is large enough, where $\bar{X}$ is the mean of $X_1,\ X_2,\ \ldots\ ,\ X_{2n+1}$.

[You may assume that Y and $\bar{X}$ are normally distributed for large n.]